THREE-DIMENSIONAL STEADY STATE TRAVELING WAVES ON A VERTICALLY FLOWING LIQUID FILM

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It is known that the character of flow of a liquid film along a vertical surface has a wave nature even at low Reynolds numbers. This is true because the flow of a film with a planar free surface is unstable [1]. In experiment two-dimensional waves are formed at some distance from the point of liquid escape. These waves grow rapidly in amplitudes and exit into a regime in which the phase velocity of the waves, their amplitude and length remain practically constant. However the length of the two-dimensional steady state traveling wave zone is usually small — further along the flow these waves break up and become three-dimensional [2]. Use of artificially imposed perturbations significantly expands the range of wave numbers at which two-dimensional regular waves are possible, although such waves also become three-dimensional, albeit significantly further from the entrance than in the case of natural flow [2, 3].

Theoretical treatment of wave flow of a liquid film in a complete formulation is extremely complex, since it becomes necessary to solve a system of nonlinear Navier–Stokes equations with a free boundary not known beforehand.

Therefore various simplifications are used for the solution. Thus, study of two-dimensional longwave perturbations for moderate flow rates can be reduced to solution of a system for two instantaneous values of the film thickness h and flow rate q [4]:

$$\frac{\partial q}{\partial t}+1{,}2\frac{\partial}{\partial x}(q^2/h)=-3\nu q/h^2+gh+\frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial x^3},\qquad \frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0. \tag{1}$$

where ν is the kinematic viscosity coefficient, g is the acceleration of gravity, and σ is the surface tension coefficient.

System (1) is obtained by integration of the equations of motion over the y-coordinate, perpendicular to the flow plane xz, and use of the assumption of self-similarity of the lognitudinal velocity component profile

$$u=1{,}5q/h(2y/h-y^2/h^2). \tag{2}$$

Nonsteady state solutions of this system which agree well quantitatively with experiment were constructed in [5, 6], and steady state traveling waves were found in [7-10].

By specifying together with Eq. (2) the velocity component profile in the z-direction in the form of a second degree polynomial

$$w=1{,}5Q/h(2y/h-y^2/h^2), \tag{3}$$

[11] integrated the system of Navier–Stokes equations to obtain a system of equations which generalizes Eq. (1) to the case of three-dimensional perturbations:

$$\frac{\partial q}{\partial t}+1{,}2\left(\frac{\partial}{\partial x}\frac{q^2}{h}+\frac{\partial}{\partial z}\frac{qQ}{h}\right)=-3\nu q/h^2+gh+\frac{\sigma h}{\rho}\left(\frac{\partial^3 h}{\partial x^3}+\frac{\partial^3 h}{\partial x\partial z^2}\right),$$
$$\frac{\partial Q}{\partial t}+1{,}2\left(\frac{\partial}{\partial x}\frac{qQ}{h}+\frac{\partial}{\partial z}\frac{Q^2}{h}\right)=-3\nu Q/h^2+\frac{\sigma h}{\rho}\left(\frac{\partial^3 h}{\partial z^3}+\frac{\partial^3 h}{\partial z\partial x^2}\right), \tag{4}$$
$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}+\frac{\partial Q}{\partial z}=0,$$

where Q is the instantaneous flow rate in the z-direction, with the remaining quantities being the same as in Eq. (1).

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We note that although assumption (3) is less obvious than Eq. (1) in the planar case, and has not had the analogous experimental [12] and theoretical [13] confirmation, still for perturbations long enough in the z-direction it will not lead to results incorrect in principle and can be used as a model.

In [11], using eight terms of a dual Fourier series for base functions in the Galerkin method nonsteady state solutions of system (4) periodic in x and z were considered. The goal of the present study is to find steady state traveling waves.

With the aid of the replacements $\bar{h} = h/h_0$, $\bar{q} = q/q_0$, $\bar{Q} = Q/q_0$, $\bar{x} = (\mathrm{We}/3)^{1/2}x/h_0$, $\bar{z} = (\mathrm{We}/3)^{1/2}z/h_0$, $\bar{t} = (\mathrm{We}/3)^{1/2}h_0^2 t/q_0$, $p = (3\mathrm{We})^{1/2}/\mathrm{Re}$, $F = (\mathrm{We}/3)^{1/2}/\mathrm{Fr}$ (where q_0 is the mean wave flow rate along the x-axis, and h_0 is the mean film thickness, $\mathrm{We} = \sigma h_0/\rho q_0^2$, $\mathrm{Re} = q_0/\nu$, $\mathrm{Fr} = q_0^2/g h_0^3$), omitting the dedimensionalization symbol, we rewrite Eq. (4) in the form

$$\frac{\partial q}{\partial t} + 1{,}2\left(\frac{\partial}{\partial x}\frac{q^2}{h} + \frac{\partial}{\partial z}\frac{qQ}{h}\right) = -pq/h^2 + Fh + 3h\left(\frac{\partial^3 h}{\partial x^3} + \frac{\partial^3 h}{\partial x \partial z^2}\right),$$
$$\frac{\partial Q}{\partial t} + 1{,}2\left(\frac{\partial}{\partial x}\frac{qQ}{h} + \frac{\partial}{\partial z}\frac{Q^2}{h}\right) = -pQ/h^2 + 3h\left(\frac{\partial^3 h}{\partial z^3} + \frac{\partial^3 h}{\partial z \partial x^2}\right), \quad (5)$$
$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial Q}{\partial z} = 0.$$

Linearizing Eq. (5) and representing the solution in the form

$$h_1 = A \exp\left[i\alpha(x - ct) + i\beta z\right] + \text{c.c.},$$
$$q_1 = B_1 A \exp\left[i\alpha(x - ct) + i\beta z\right] + \text{c.c.}, \quad (6)$$
$$Q_1 = D_1 A \exp\left[i\alpha(x - ct) + i\beta z\right] + \text{c.c.}$$

(where c.c. denotes the complex conjugate), it is simple to show that perturbations with wave numbers α and β lying within (without) the semicircle of Fig. 1, curve 1,

$$(\alpha^2 + \beta^2)^2 = \alpha^2, \quad (7)$$

are unstable (stable), since they have an imaginary component $c_i > 0$ ($c_i < 0$). Perturbations with wave numbers α and β satisfying Eq. (7) are neutral, with c = 3, and the coefficients B_1 and D_1 in Eq. (6) are expressed in the following manner:

$$D_1 = 3\alpha\beta/(1{,}8\alpha + ip),\ B_1 = 3 - \beta D_1/\alpha. \quad (8)$$

The solution of Eq. (6) branches from the planoparallel h = 1, q = 1, Q = 0 to the plane of the wave numbers α, β along line (7) and is valid for the case where A is infinitely small.

To find steady state traveling waves of finite amplitude which are a solution of system (5), we transform therein to the variables ξ = x − ct, z = z:

$$-c\frac{\partial q}{\partial \xi} + 1{,}2\left(\frac{\partial}{\partial \xi}\frac{q^2}{h} + \frac{\partial}{\partial z}\frac{qQ}{h}\right) = -pq/h^2 + Fh + 3h\left(\frac{\partial^3 h}{\partial \xi^3} + \frac{\partial^3 h}{\partial \xi \partial z^2}\right),$$
$$-c\frac{\partial Q}{\partial \xi} + 1{,}2\left(\frac{\partial}{\partial \xi}\frac{qQ}{h} + \frac{\partial}{\partial z}\frac{Q^2}{h}\right) = -pQ/h^2 + 3h\left(\frac{\partial^3 h}{\partial z^3} + \frac{\partial^3 h}{\partial z \partial \xi^2}\right), \quad (9)$$
$$-c\frac{\partial h}{\partial \xi} + \frac{\partial q}{\partial \xi} + \frac{\partial Q}{\partial z} = 0.$$

For the future we will limit our search to solutions of system (9) (periodic in ξ and z) for which h is a function symmetric with respect to z.

Since the amplitude of perturbations which are neutral in linear stability theory are infinitely small, it is clear that the amplitude of steady state waves with wave numbers lying near curve (7) will be finite, but small. Therefore for such a wave number we will seek a solution of Eq. (9) in the form of a series in the small parameter:

$$h = 1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots,\ q = 1 + \varepsilon q_1 + \varepsilon^2 q_2 + \ldots,$$
$$Q = \varepsilon Q_1 + \varepsilon^2 Q_2 + \ldots,$$
$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots,\ c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \ldots$$

Following [14], we introduce a set of rapid and slow variables:

$$\xi_0 = \xi,\ \xi_n = \varepsilon^n \xi,\ z_0 = z,\ z_n = \varepsilon^n z,\ n = 1, 2, \ldots$$

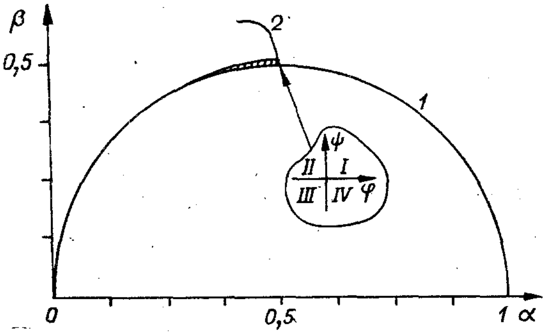

Fig. 1

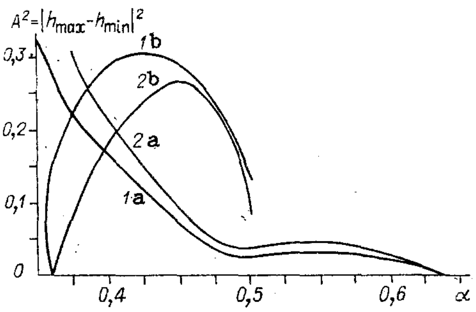

Fig. 2

From the zeroth approximation in ε we obtain $F_0 = p$. From Eq. (9), for the first approximation we have a linear system with solution in the form

$$\begin{aligned} h_1 &= Ae^{i\alpha\xi_0}\left(ae^{i\beta z_0} + \bar{a}e^{-i\beta z_0}\right) + \text{c.c.}, \\ q_1 &= B_1Ae^{i\alpha\xi_0}\left(ae^{i\beta z_0} + \bar{a}e^{-i\beta z_0}\right) + \text{c.c.}, \\ Q_1 &= D_1Ae^{i\alpha\xi_0}\left(ae^{i\beta z_0} - \bar{a}e^{-i\beta z_0}\right) + \text{c.c.}, \end{aligned} \tag{10}$$

where α, β, B_1, D_1 satisfy Eqs. (7), (8); $c_0 = 3$.

It follows from the symmetry of the unknown solutions that A can be a function only of $\xi_1, \xi_2, \ldots$, while a depends on $z_1, z_2, \ldots$. Determination of their form by consideration of higher approximations permits determination of the relationship between corrections to the wave numbers α and β and the value of the amplitude of the first harmonic. But, for example, knowing the correction to β − Δβ, one can simply say that the solution is near a new point on the curve of Eq. (7), at which $\beta_1 = \beta + \Delta\beta$. Therefore in Eq. (10) we assume a = const and include its value in A; in other words, we will seek a solution of finite amplitude, rigidly fixing the value of the wave number β. This can be done almost everywhere on the curve of Eq. (7), with the exception of the immediate vicinities of several separate points.

For the parameter ε we may choose the value of A. It can easily be shown that from the requirement of absence of secular terms in the second approximation

$$c_1 = 0,\ \partial A/\partial\xi_1 = 0,\ F_1 = 0.$$

With consideration of this, for the second approximation we arrive at the system

$$\begin{gathered} -c_0\frac{\partial q_2}{\partial \xi_0} + 1{,}2\left[\frac{\partial}{\partial \xi_0}\left(2q_2 - h_2 + (q_1 - h_1)^2\right) + \frac{\partial}{\partial z_0}\left(Q_2 + Q_1(q_1 - h_1)\right)\right] = -p\left(q_2 - 3h_2 + 3h_1^2 - 2q_1h_1\right) + F_2 + \\ + 3\left(\frac{\partial^3 h_2}{\partial \xi_0^3} + h_1\frac{\partial^3 h_1}{\partial \xi_0^3} + \frac{\partial^3 h_2}{\partial \xi_0 \partial z_0^2} + h_1\frac{\partial^3 h_1}{\partial \xi_0 \partial z_0^2}\right), \\ -c_0\frac{\partial Q_2}{\partial \xi_0} + 1{,}2\left[\frac{\partial}{\partial \xi_0}\left(Q_2 + Q_1(q_1 - h_1)\right) + \frac{\partial Q_1^2}{\partial z_0}\right] = -p\left(Q_2 - 2h_1Q_1\right) + 3\left(\frac{\partial^3 h_2}{\partial z_0^3} + h_1\frac{\partial^3 h_1}{\partial z_0^3} + \frac{\partial^3 h_2}{\partial z_0 \partial \xi_0^2} + h_1\frac{\partial^3 h_1}{\partial z_0 \partial \xi_0^2}\right), \\ -c_0\frac{\partial h_2}{\partial \xi_0} + \frac{\partial q_2}{\partial \xi_0} + \frac{\partial Q_2}{\partial z_0} = 0, \end{gathered} \tag{11}$$

solving which, we find

$$\begin{aligned} h_2 &= A_2A^2e^{i2\alpha\xi_0}\left(e^{i2\beta z_0} + e^{-i2\beta z_0}\right) + A_3|A|^2\left(e^{i2\beta z_0} + e^{-i2\beta z_0}\right) + A_4A^2e^{i2\alpha\xi_0} + \text{c.c.}, \\ q_2 &= B_2A^2e^{i2\alpha\xi_0}\left(e^{i2\beta z_0} + e^{-i2\beta z_0}\right) + B_3|A|^2\left(e^{i2\beta z_0} + e^{-i2\beta z_0}\right) + B_4A^2e^{i2\alpha\xi_0} + \text{c.c.}, \\ Q_2 &= D_2A^2e^{i2\alpha\xi_0}\left(e^{i2\beta z_0} - e^{-i2\beta z_0}\right) + \text{c.c.}, \end{aligned} \tag{12}$$

where

$$\begin{gathered} A_2 = -[1{,}2(\alpha(B_1 - 1) + \beta D_1)^2 + 1{,}5\alpha^2 + 1{,}5i\alpha p]/9\alpha^2, \\ D_2 = [-24i\alpha\beta A_2 - 2{,}4i(\alpha(B_1 - 1) + \beta D_1)D_1 + 2pD_1 - 3i\alpha\beta]/(p - 3{,}6i\alpha), \\ B_2 = c_0A_2 - \beta D_2/\alpha, \\ A_3 = (2{,}4\beta|D_1|^2 - ip(D_1 - \bar{D}_1) - 3\alpha\beta)/24\beta^3, \\ B_3 = 3A_3 + 3 - 25{,}2\alpha\beta^2/(p^2 + 3{,}24\alpha^2), \end{gathered}$$

$$A_4 = (2{,}4\alpha(B_1 - 1)^2 + 3\alpha^2 - ip(3 - 2B_1))/3\alpha(1 - 4\alpha^2), \quad B_4 = c_0 A_4. \tag{13}$$

Since for the dedimensionalization of Eq. (4) we chose as characteristic scales the mean wave flow rate q_0 and the mean thickness h_0, from the condition of absence of secular terms, from the first equation in Eq. (11) we have

$$F_2 = 4p(3 - B_1 - \overline{B_1})|A|^2.$$

From the condition of absence of secular terms in the third approximation we obtain

$$\begin{aligned} &6i\alpha(1-2\alpha)\frac{\partial A}{\partial \xi_2} + \{p\,[12i\beta(D_1+\overline{D}_1) - 45i\alpha + 6i\alpha(A_2 + A_4 + 2A_3) - \\ &- 4i\beta(\overline{D}_1 A_4 + 2D_1 A_3)] + 9{,}6\,[\alpha^2(B_3 - A_3 + A_2 + A_4 - 4{,}5) - \beta^2|D_1|^2 - \\ &- \alpha\beta(\overline{D}_1 A_4 - D_1(A_3 - B_3 - 3) + \overline{D}_1)] + (6\alpha^4 + 42\beta^4)A_3 + 21\alpha^2 A_2 + \\ &+ (21\alpha^4 + 3\beta^4)A_4\}|A|^2 A - c_2(3{,}6\alpha^2 + i\alpha p)A = 0, \end{aligned} \tag{14}$$

Representing the solution of Eq. (14) in the form $A = A_0 e^{i\varphi\xi_2}$ and separating the imaginary and real components, we find expressions for the corrections to the wave number α and the phase

$$\varphi = f_1(\alpha, \beta, p)|A_0|^2, \; c_2 = f_2(\alpha, \beta, p)|A_0|^2. \tag{15}$$

We will not present the explicit forms of the functions f_1 and f_2 because of their cumbersomeness.

Analyzing Eqs. (13)-(15), it is simple to show that for sufficiently small values of $|A_0|$ the terms of the second approximation are small in comparison to those of the first for all α and β lying on the curve of Eq. (7), with the exception of the vicinities of the points $\alpha = 1$, $\beta = 0$; $\alpha = 0$, $\beta = 0$; $\alpha = \beta = 0.5$. In fact, it is evident from Eq. (13) that upon approach to these points some of the coefficients in Eq. (12) increase without limit, so that the solutions obtained are inapplicable. These points require special examination. For the remaining points of Eq. (7) calculations with Eq. (15) show that if $0.5 < \alpha < 1$, then for any values of p $c_2 < 0$, $\varphi < 0$. The latter means that the wave regime branches into a region of linear instability — a soft type of branching. For perturbations having α in the interval (0; 0.5) the situation is more complex — for sufficiently high values of $p \gtrsim 1$ the correction φ over the entire interval is negative. This indicates that for such α the wave vectors of steady state traveling waves of low but finite amplitude lie outside the instability region. With decrease in p (which corresponds to growth in Re) the value of the negative correction to the modulus decreases, and at $p^* = 0.81158$ for $\alpha^* = 0.29$ (correspondingly $\beta^* = 0.454$) it vanishes. With further decrease in p an interval of α appears and increases in size, in which $\varphi > 0$, i.e., branching into the region of instability occurs again. The correction to the phase velocity in the vicinity of $\alpha \approx 0.5$ (but with $\alpha < 0.5$) for any value of p is always positive. But with motion along the arc of Eq. (7) in the direction of lower α the value of c_2 decreases, passing through zero at the point α_+. With increase in p α_+ decreases, and as $p \to 0$, $\alpha_+ \to 0.5$.

With approach to the point $\alpha = 0$, $\beta = 0$ along the curve of Eq. (7) for any finite value of $p < p^*$, beginning with $\alpha_* < \alpha^*$, periodic weakly nonlinear regimes again branch outward from the instability region. The value of α_* falls with decrease in p. In the vicinity of the point $\alpha = 0$, $\beta = 0$ wave regimes with amplitudes ~ε are possible, if the components of the wave vector are of the order of smallness $\beta \sim \varepsilon$, $\alpha \sim \varepsilon^2$. Moreover, it is necessary that the components of the wave vector at which the change in the type of branching occurs be of the same order of magnitude. The latter is possible if the order of smallness of the parameter $p \gtrsim \varepsilon^2$.

In obtaining a solution in the vicinity of the point $\alpha = 1$, $\beta = 0$ it must be considered that for neutral perturbations with components of the wave vector lying on the curve of Eq. (7) (7), at $\beta \ll 1$, $\alpha \approx 1 - \beta^2$. Correspondingly, the correction to α when finding a solution with amplitude ~ε must be of a greater order of smallness than for β. Therefore in the expression for the first approximation we may write

$$h_1 = Ae^{i\xi_0} + \text{c.c.}, \; q_1 = 3h_1, \; Q_1 = 0,$$

where A is a function of the slow coordinates $z_1, \xi_2, \ldots$.

The second approximation has the form

$$h_2 = A_2A^2 e^{i2\xi_0} + \text{c.c.},\ q_2 = 3h_2,\ A_2 = -0{,}7 - ip/6,\quad Q_2 = B_2 \frac{\partial A}{\partial z_1} e^{i\xi_0} + \text{c.c.},\ B_2 = -3/(p - 1{,}8i),$$

and the requirement of absence of secular terms in the third approximation leads to the equation

$$c_2 (3{,}6 + ip) A + 6 \frac{\partial^2 A}{\partial z_1^2} + 6i \frac{\partial A}{\partial \xi_2} = (p^2 - 35{,}82 - 30{,}3ip) |A|^2 A.$$

Using the representation $A = A_0 e^{i\varphi\xi_2} (e^{i\psi z_1} + e^{-i\psi z_1})$, we obtain

$$c_2 = -60{,}6 |A_0|^2,\ \psi^2 + \varphi = -(p^2 + 73{,}26)|A_0|^2/3. \tag{16}$$

From this it follows that as for all $\alpha > 0.5$ in the vicinity of the point $\alpha = 1$, $\beta = 0$ the weakly nonlinear wave regimes branch into the instability region and their phase velocity is less than $c_0 = 3$. At $\psi = 0$ Eq. (16) defines the correction values for plane waves.

In finding the solution in the vicinity of $\alpha = 0.5$, $\beta = 0.5$ we note that [since for a wave with this wave vector one of the even harmonics ($\alpha = 1$, $\beta = 0$) has a wave vector also lying on the neutral curve of Eq. (7)] in the first approximation in place of Eq. (10) we have

$$h_1 = A e^{i\xi/2} (a e^{iz/2} + \bar{a} e^{-iz/2}) + N e^{i\xi} + \text{c.c.},$$
$$q_1 = B_1 A e^{i\xi/2} (a e^{iz/2} + \bar{a} e^{-iz/2}) + 3N e^{i\xi} + \text{c.c.},$$
$$Q_1 = D_1 A e^{i\xi/2} (a e^{iz/2} - \bar{a} e^{-iz/2}) + \text{c.c.},$$

Here A and N are slowly changing functions of the variable ξ, and a is a function of z. Generally speaking, aA and N may be of different orders of smallness. Using the deviations of the wave number components from the point (0.5, 0.5) as small parameters, in analogy to [15] we seek a solution in the form

$$A = A_0 e^{i\varphi\xi},\ a = a_0 e^{i\psi z},\ N = N_0 e^{i2\varphi\xi}.$$

The values of A_0 and a_0 can be considered real without limiting generality. Setting the secular terms in the subsequent approximation equal to zero, after performance of simple but cumbersome calculations to determine A_0, a_0, N_0, and the corrections to the phase velocity c_1 we find the system

$$(6\varphi - c_1(3{,}6 + ip))N_0 + (2{,}4(2 - D_1)^2 + 1{,}5 + 2ip(3 - 2D_1)) \times (A_0 a_0)^2 = 0,$$
$$6\psi + 3\varphi^2 - c_1(1{,}8 + ip) + (4{,}8(1 - \bar{D}_1) + 3 + 2ip(3 - 2\bar{D}_1))N_0 = 0, \tag{17}$$

solving which, we see that in the wave number plane in regions where $\varphi\psi < 0$ (quadrants II and IV of a coordinate system with center at point $\alpha = \beta = 0.5$ and axes directed in the direction of increase of α and β, see Fig. 1) two solutions exist. For one of these the value of the correction to the phase velocity c_0 $c_1 > 0$, while for the other $c_1 < 0$. Quadrants I and III have one solution each, with $c_1 > 0$ and $c_1 < 0$, respectively.

In comparing results of calculations with Eqs. (15) and (17), it becomes clear that there are at least two types of weakly nonlinear solutions of system (9). The region in which the first type exists for $\alpha \geq 0.5$ lies below the curve of Eq. (7), and in the vicinity of $\alpha = \beta = 0.5$ occupies quadrants II and IV. The solution with correction to the phase velocity $c_1 < 0$, which branches weakly from a planoparallel state for $\alpha > 0.5$ along the curve of Eq. (7), passing below the point $\alpha = \beta = 0.5$ with continuous change in wave numbers, and at $\alpha < 0.5$ exiting into a linear stability region with finite amplitude on the curve of Eq. (7). As $\alpha \to 0.5$, $\beta \to 0.5$ with $\alpha < 0.5$, $\beta > 0.5$ this solution disappears. In the future we will speak of such wave regimes as type I waves.

Solutions of the second type branch from planoparallel along the curve of Eq. (7) for $\alpha < 0.5$ everywhere at $p > p^*$ into the linear stability region, while for $p < p^*$ there exists an interval of α values with soft type branching ($\alpha_* < \alpha < \alpha_{**} < 0.5$). The region of existence of these solutions in the vicinity of the point $\alpha = \beta = 0.5$ includes quadrants II, I, and IV.

To find wave regimes with wave numbers differing markedly from neutral, system (9) was solved numerically. The solution was sought in the form of finite sums of double Fourier series

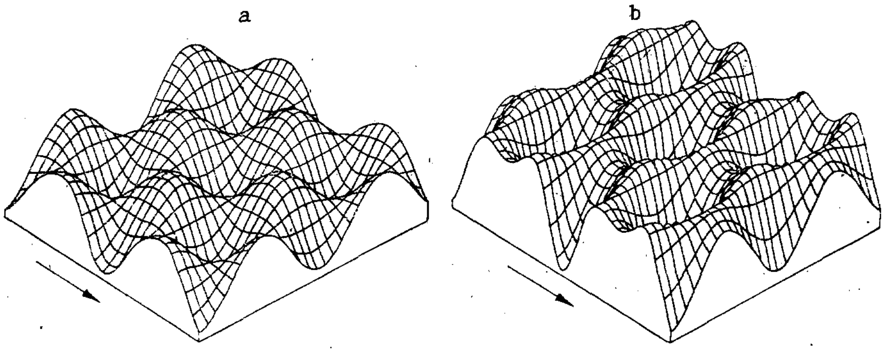

Fig. 3

$$h = \sum_{m=-M}^{M} \sum_{n=-N}^{N} H_{mn} e^{i\alpha m\xi + i\beta nz}, \quad q = \sum_{m=-M}^{M} \sum_{n=-N}^{N} q_{mn} e^{i\alpha m\xi + i\beta nz}. \tag{18}$$

In obtaining the majority of the above results it was assumed that M = N = 3, although for some parameter values calculations were performed with M = 7, N = 3. The method used for solution involved Fourier transformation and was similar to that used in [8, 10] for plane waves. Initially solutions were sought near the neutral curve, where the analytical results work well as an initial approximation. Movement over wave numbers away from the curve of Eq. (7) was accomplished continuously. Below we present results obtained for

$$p = 1 > p^*, \ p = 0{,}5 < p^*.$$

Figure 2 shows the square of the amplitude of steady state waves $A^2 = |h_{\max} - h_{\min}|^2$ as a function of wave number α for a horizontal section of the linear instability region for the level β = 0.48. Lines 1, 2 correspond to p = 1, 0.5, while the indices a, b indicate waves of the first and second types.

For α > 0.5 the characteristic form of a type I wave is shown in Fig. 3a, where p = 0.5, F = 0.498, α = 0.55, β = 0.48, c = 2.923, A = 0.212. For α ≲ 0.5 these waves become more "nonlinear" – their amplitude, as is obvious, begins to increase rapidly. Their typical form is shown in Fig. 3b, where p = 0.5, F = 0.492, α = 0.45, β = 0.48, c = 2.772, A = 0.302.

For these values of α with further decrease the harmonic H_{20} increases rapidly as compared to others. This is apparently related to the fact that having entered the linear instability region after passage through the line α = 0.5, with decrease in the value of the wave number α the harmonic H_{20} at fixed β is the only one which continues to penetrate further into this region.

To calculate waves of this type with α ≲ 0.4 a large number of harmonics are required. Thus, for the variants calculated with M = N = 3 and M = 7, N = 3 in the sums of Eq. (18) for 0.5 ≲ α ≲ 0.64, the difference in thicknesses and wave velocities comprises ~1%, while for α = 0.4 the solutions differ by ~10%, although their qualitative behavior remains the same.

As is evident from Fig. 2, type II waves at values of the parameter p = 1 (curve 1b) branch into the stability region, while at p = 0.5 (curve 2b) there is a soft branching regime [here α_n = 0.36 lies on the neutral curve of Eq. (7)]. These results are in complete agreement with those obtained analytically. Thus, it follows from Eq. (15) that for p = 0.5 the change in the type of branching occurs at the point α_{**} = 0.4289 on the curve of Eq. (7). From the calculations we find that α_{**} lies in the range 0.425-0.423.

In Fig. 1 within the linear stability region a zone is shaded, in which type II waves were found for p = 1. Although as follows from Eq. (15) such a zone also exists for lower values of α, it then practically merges with the neutral curve. In accordance with results obtained by solving Eq. (17), the right-hand boundary goes beyond the line α = 0.5, but by only such a small distance (α < 0.503) that it is not visible in the scale used. The analogous zone for p = 0.5 is still narrower, and is limited on the left by the line $\alpha = \alpha_{**}$ = 0.4289.

The characteristic form of a type II wave is shown in Fig. 4. Here p = 0.5, F = 0.485, α = 0.475, β = 0.48, c = 2.975, A = 0.481.

Type II waves move more rapidly than type I. Thus, while for the latter the phase velocity values are always less than c_0 = 3 and with removal from the branching point [α > 0.5

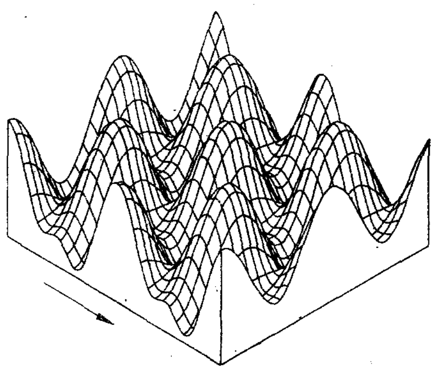

Fig. 4

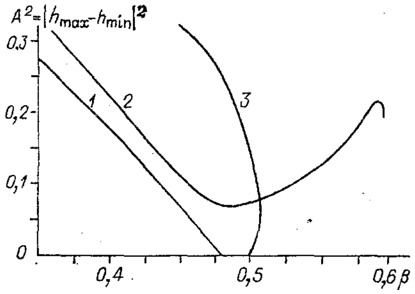

Fig. 5

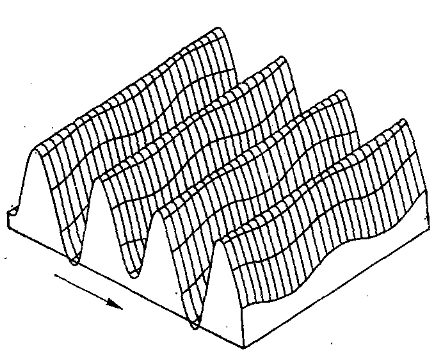

Fig. 6

on the curve of Eq. (7)] differ markedly therefrom (up to 20-25% for the parameter values considered), for type II waves the difference from c_0 does not exceed 1-1.5% to one or the other side.

Figure 5 shows the dependence of A^2 on the wave number for several vertical sections α = const, with curve 1 showing how the amplitude of a type I wave in the section $\alpha = 0.64$ varies with motion from the branching point ($\beta_n = 0.48$) into the depths of the instability region. It is evident that up to $\beta \sim 0.35$ the amplitude increases $\sim(\beta - \beta_n)^{1/2}$ [a dependence analogous to Eq. (15)]. It is clear that these waves are quite close to weakly nonlinear (for $\alpha > 0.5$). This is also indicated by their form (see Fig. 3) — almost sinusoidal behavior in the ξ- and z-directions.

Similar relationships for the same type of wave in the section $\alpha = 0.45$ are shown by curve 2. It is evident that with immersion into the instability region the amplitude also increases. Upon exit from the linear instability region ($\beta > \beta_n = 0.4975$) the harmonic H_{20} increases rapidly, producing the dominant contribution to A. This leads to a situation in which the spatial waves begin to degenerate into weak plane waves modulated in the z-direction (Fig. 6, where p = 0.5, F = 0.481, $\alpha = 0.45$, $\beta = 0.532$, c = 2.813, A = 0.314), which become planar at some $\beta_* = \beta_*(\alpha, p)$. Thus in the section $\alpha = 0.45$ for p = 1 $\beta_* = 0.594$, for p = 0.5 $\beta_* = 0.533$. Control calculations with M = 7, N = 3 in Eq. (18) indicated that the change in β_* does not exceed 3%. The function $\beta_* = \beta_*(\alpha, p)$ for p = 1 is shown by line 2 of Fig. 1, which requires a larger number of harmonics for extension into the region with $\alpha < 0.4$.

Similar degeneration of spatial solutions into plane waves with double wave number was observed in [15] for an equation describing behavior of perturbations on a film for Re $\lesssim$ 1.

The behavior of a type II wave for $\alpha = 0.47$ is shown by curve 3 of Fig. 5. For motion into the depths of the instability region, in contrast to type I waves the "high" harmonics H_{3n} increase much more rapidly, and even for calculation of waves with $\beta \lesssim 0.45$ the number of harmonics must be increased. It can thus be said that type II waves are more nonlinear than type I waves.

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FLEXURAL PERTURBATIONS OF FREE JETS OF MAXWELL AND DOI–EDWARDS LIQUIDS

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UDC 532.522+532.135

Flexural perturbations of high-velocity free jets of drop liquids moving in air are reinforced by the fact that the air pressure on the concave sections of the jet surface is greater than on the convex sections. The linear and nonlinear stages of development of flexural perturbations were studied in [1-5] for viscous Newtonian fluids. The effect of elastic stresses in the fluid on the growth of flexural perturbations of jets was first examined in [6], where it was assumed in an analysis of the growth of small disturbances that surface tension was constant along the jet, i.e., the investigators actually studied a tensed string. The studies [7, 8] examined the linear stage of growth of flexural perturbations of jets of Maxwell liquids. Our goal here is to analyze the dynamics of long-wave flexural perturbations of jets of viscoelastic fluids in both the linear and nonlinear stages of development. The rheological behavior of the fluid is described by two models – the phenomenological (Maxwell) model and the physical-molecular (Doi–Edwards) model. It is shown that the disturbances are oscillatory in character in the nonlinear stage of development. Meanwhile, the results of calculations performed with the Maxwell (M) and Doi–Edwards (DE) rheological models in the given problem agree with each other quantitatively as well as qualitatively.

1. We will examine a free jet of a drop liquid moving at the velocity U_0 in air. In the undisturbed state, the axis of the jet is straight, while its cross section is a circle of radius a_0. The densities of the liquid and air will be denoted by ρ and ρ_1, while the surface tension of the liquid will be denoted by α. The liquid is assumed to be viscoelastic. As usual, the relationship between the deviator of the stress tensor σ' and the kinematic and geometric parameters is determined by the rheological equation of state. From among rheological equations of state for concentrated systems in the literature, we will choose the two with the clearest physical meaning. The first, the Maxwell rheological equation, determines the deviator of the stress tensor in the form [9]

$$\boldsymbol{\sigma}'(t)=\frac{\mu}{\theta^2}\int\limits_{-\infty}^{t} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{B}_\tau(t)-\mathbf{g}]_\mathbf{s} \quad (1.1)$$

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